Life expectancy, average age of death, and tempo-adjustment.*

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Abstract

The recent proposal by Bongaarts and Feeney for a tempo-adjustment to life expectancy is assessed using numerical techniques. I find that for a wide variety of situations no such adjustment is necessary provided mortality is observed in a given period. Bongaarts and Feeney assume the force of mortality affecting the mean age of death is not always observed. The present work brings into sharper relief the question of why this aspect of mortality is not observed, a claim of Bongaarts and Feeney not adequately justified in their paper.

Studies of mortality have been at the center of formal demography since Graunt's (1662) *Bills of Mortality* for London (cf. especially chapter XI, §9–11) and Halley's (1693a, b) life table for Breslau, Silesia (now Wrocław, Poland). Life expectancy has long been a basic ingredient of mortality analysis, and is probably the most sought-after demographic statistic by non-demographers. John Bongaarts and Griffith Feeney (2002) recently proposed an intriguing tempo-adjustment for life expectancy. Their paper is interesting and important, and will potentially

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redefine the way demographers think about mortality change. However, in this comment I present numerical analysis that suggests some further refinements to their approach are warranted.

Life expectancy and average age at death

Bongaarts and Feeney postulate that when mortality is changing, the equivalence between life expectancy and the average age at death does not hold, especially at low levels of mortality. They propose a correction factor that produces a tempo-adjusted mean age at death. Life expectancy data missing this correction factor are misleading statistics according to Bongaarts and Feeney. I begin as Bongaarts and Feeney do, with a Gompertizian model for the force of mortality, modified to exclude mortality below age 30:

$$\mu(a,t) = \begin{cases} 0 & 0 \le a < 30 \\ \mu(0,t)\exp(ba) & 30 \le a < \omega, \end{cases} \tag{1}$$

where $b > 0$ is a parameter and $\mu(0,t)$ is a baseline value. Life expectancy is:

$$e_0(t) = \int_0^{\omega} \exp\left(-\int_0^{a} \mu(x,t)dx\right) da = 30 + \int_{30}^{\omega} \exp\left(-\int_{30}^{a} \mu(0,t)\exp(bx)dx\right) da. \tag{2}$$

The limits of integration in (2) change because $\mu(\cdot,t)$ vanishes below age 30.

Bongaarts and Feeney also define starred versions of these relations for mortality shifting over time. The tempo-adjusted mean age at death, $e_0^*(t)$, is expressed in terms of a differential equation:

$$e_0^*(t) = e_0(t) + \frac{1}{b}\log\left(1 - \frac{de_0^*(t)}{dt}\right). \tag{3}$$

And the starred force of mortality is:

$$\mu^*(a,t) = \begin{cases} 0 & 0 \le a < 30 \\ \mu(0,t)\left(1 - \frac{de_0^*(t)}{dt}\right)^{-1} \exp(ba) & 30 \le a < \omega. \end{cases} \tag{4}$$

Equations (3) and (4) hold when $\frac{d}{dt}e_0^*(t) < 1$, which is not an unreasonable assumption for unadjusted $e_0(t)$ (Oeppen and Vaupel, 2002). Steeper rates of improvement of tempo-adjusted mean age at death would have to be analyzed some other way.

What, Bongaarts and Feeney ask, is the average age at death for a population experiencing death rates changing over time? This is their key question. The answer is (from Appendix B of Bongaarts and Feeney):

$$A(t) = \frac{\int_0^{\omega} a\mu^*(a,t)\ell^*(a,t)da}{\int_0^{\omega} \mu^*(a,t)\ell^*(a,t)da} = \int_0^{\omega} a\mu^*(a,t)\exp\left(-\int_0^{a}\mu^*(x,t)dx\right)da. \tag{5}$$

The denominator in the first instance may be shown to equal exactly one, simplifying (5) to the double integral in the right hand side. Bongaarts and Feeney assert: $A(t) = e_0^*(t)$.

Consider again life expectancy, $\int_0^{\omega} \ell(a)da$. As defined in (2), $e_0(t)$ applies to the mortality regime in (1). Bongaarts and Feeney assert that when $\mu(\cdot,t)$ changes to $\mu^*(\cdot,t)$ that life expectancy no longer measures the mean age at death, which becomes $e_0^*(t) = e_0(t) + b^{-1}\log(1 - \frac{de_0^*(t)}{dt})$. *The full version of the question ought to be:* Which *life expectancy does not equal the mean age of death?* That is to say, if mortality rates change, so does life expectancy as any demographer would calculate it. Specifically, for mortality as in (4):

$$e_0^\dagger(t) = \int_0^\omega \exp\left(-\int_0^a \mu^*(x,t)dx\right) da = 30 + \int_{30}^\omega \exp\left(-\int_{30}^a \frac{\mu(0,t)}{1-\frac{de_0^*(t)}{dt}} \exp(bx)dx\right) da. \tag{6}$$

Thus there are three candidates for the mean age at death:

$$A(t) \stackrel{?}{=} \begin{cases} 30 + \int_{30}^\omega \exp\left(-\int_{30}^a \mu(0,t)\exp(bx)dx\right) da + \frac{1}{b}\log\left(1-\frac{de_0^*(t)}{dt}\right) & \{\text{B. \& F.}\} \\ 30 + \int_{30}^\omega \exp\left(-\int_{30}^a \frac{\mu(0,t)}{1-\frac{de_0^*(t)}{dt}} \exp(bx)dx\right) da & \{e_0^\dagger(t)\} \\ \int_{30}^\omega a \frac{\mu(0,t)}{1-\frac{de_0^*(t)}{dt}} \exp\left(ba - \int_{30}^a \frac{\mu(0,t)}{1-\frac{de_0^*(t)}{dt}} \exp(bx)dx\right) da & \{\text{1st moment}\} \end{cases} \tag{7}$$

How to adjudicate among the three forms in (7)? The 'gold standard' for the mean age at death is the third, i.e., (5), which is the the first moment of the distribution of ages of death—the textbook statistical definition of the mean. Between the other two formulations, which is closest to the true mean age of death?

Integrals of double exponentials have no solution in terms of elementary functions. Solving (7) analytically involves recourse to the 'special function' Ei, the exponential integral (cf. Widder 1961:ch. 13, §6.1). In short, further manipulation of (7) is not a route to neat analytic relationships. However, since the integrals are definite, they may be evaluated to as many digits of precision as desired; Press et al. (1992, §6.3) discuss the particulars.

I used the computer algebra program Maple (Waterloo Maple, Inc., 2000) to solve the integrals of (7) numerically. Numerical integration is one of the most tried and tested computer techniques, and is very precise. Table 1 presents a sampling of the results, in the form of a $4 \times 3 \times 4$ contingency table, for several possible values of b, $\mu(0,t)$ and $\frac{d}{dt}e_0^*(t)$. The oldest age to which anyone lives, ω, was assumed to be 150 years. For each parameter combination I calculate: Bongaarts and Feeney's $e_0^*(t)$; $e_0^\dagger(t)$; and the mean age at death (first moment). Life expectancy calculated in the usual way but using $\mu^*(\cdot,t)$ rather than $\mu(\cdot,t)$ is $e_0^\dagger(t)$, and it matches the first moment to at least the fifth decimal place in all cases. Most cases match to seven decimal places, and some $e_0^\dagger(t)$ values are digit-exactly the same as the first moment. On the other hand, Bongaarts and Feeney's $e_0^*(t)$ matches the first moment only to the first or second decimal place (except when $\frac{d}{dt}e_0^*(t) = 0$, since $e_0^*(t) \equiv e_0^\dagger(t)$ in this case).

How does this inconsistency arise? The reader may verify that the top formula in (7), i.e. my (3), is exactly equation (12) of Bongaarts and Feeney (p. 21). The bottom formula in (7), i.e. my (5), is exactly equation (16b) of Bongaarts and Feeney (p. 28). Bongaarts and Feeney assert that these two forms are equal, which is not the case for the data in table 1. The middle equation of (7), the standard life expectancy for the mortality schedule in (4), is a far superior match to the moment-based mean age of death.

Bongaarts and Feeney underscore that for real-world data $\mu^*(\cdot,t)$ is not observed, and therefore that the way to calculate the mean age at death is through $e_0^*(t)$ (p. 23), which converges to life expectancy when $\frac{d}{dt}e_0^*(t) = 0$. This point is not lost on me. Nonetheless, it seems a sort of mathematical legerdemain to say that $\mu^*(\cdot,t)$ is observable when it equals $\mu(\cdot,t)$ but not otherwise.

Consider the following scenario: first $\frac{d}{dt}e_0^*(t) = 0$, then life expectancy increases monotonically, so $\frac{d}{dt}e_0^*(t) > 0$, then it stabilizes, so $\frac{d}{dt}e_0^*(t) = 0$ again. Bongaarts and Feeney claim that during the middle period, $e_0(t)$ exhibits a tempo bias, and that during this period $\mu^*(\cdot,t)$, not the observed $\mu(\cdot,t)$, affects

Parameters:			Calculations:			
				B&F		
b	$\mu(0,t)$	$\frac{d}{dt}e_0^*(t)$	$b^{-1}\log\left(1-\frac{de_0^*(t)}{dt}\right)$	$e_0^*(t)$	$e_0^\dagger(t)$	1st moment
.09	$.10\times10^{-4}$	−.10	1.059001998	95.93747509	95.92764623	95.92764627
		0.	0.	94.87847309	94.87847309	94.87847311
		.10	−1.170672397	93.70780069	93.71961056	93.71961059
		.30	−3.963054932	90.91541816	90.95991701	90.95991696
	$.25\times10^{-4}$	−.10	1.059001998	85.90302375	85.88222412	85.88222411
		0.	0.	84.84402175	84.84402175	84.84402171
		.10	−1.170672397	83.67334935	83.69826873	83.69826871
		.30	−3.963054932	80.88096682	80.97448284	80.97448285
	$.40\times10^{-4}$	−.10	1.059001998	80.80995382	80.77973922	80.77973924
		0.	0.	79.75095182	79.75095182	79.75095181
		.10	−1.170672397	78.58027942	78.61641217	78.61641216
		.30	−3.963054932	75.78789689	75.92314771	75.92314770
.10	$.10\times10^{-4}$	−.10	.9531017980	87.41772272	87.40732784	87.40732784
		0.	0.	86.46462092	86.46462092	86.46462093
		.10	−1.053605157	85.41101576	85.42349888	85.42349889
		.30	−3.566749439	82.89787148	82.94487096	82.94487096
	$.25\times10^{-4}$	−.10	.9531017980	78.40929093	78.38741856	78.38741854
		0.	0.	77.45618913	77.45618913	77.45618911
		.10	−1.053605157	76.40258397	76.42876950	76.42876951
		.30	−3.566749439	73.88943969	73.98760788	73.98760789
	$.40\times10^{-4}$	−.10	.9531017980	73.84480690	73.81314852	73.81314853
		0.	0.	72.89170510	72.89170510	72.89170514
		.10	−1.053605157	71.83809994	71.87592739	71.87592742
		.30	−3.566749439	69.32495566	69.46638388	69.46638389
.11	$.10\times10^{-4}$	−.10	.8664561800	80.36009854	80.34891416	80.34891417
		0.	0.	79.49364236	79.49364236	79.49364238
		.10	−.9578228700	78.53581949	78.54924249	78.54924248
		.30	−3.242499490	76.25114287	76.30163762	76.30163769
	$.25\times10^{-4}$	−.10	.8664561800	72.19576683	72.17238274	72.17238274
		0.	0.	71.32931065	71.32931065	71.32931066
		.10	−.9578228700	70.37148778	70.39946040	70.39946041
		.30	−3.242499490	68.08681116	68.19156001	68.19156003
	$.40\times10^{-4}$	−.10	.8664561800	68.06756526	68.03385538	68.03385535
		0.	0.	67.20110908	67.20110908	67.20110909
		.10	−.9578228700	66.24328621	66.28352684	66.28352684
		.30	−3.242499490	63.95860959	64.10886245	64.10886246
.12	$.10\times10^{-4}$	−.10	.7942514983	74.41453253	74.40233176	74.40233174
		0.	0.	73.62028103	73.62028103	73.62028105
		.10	−.8780042975	72.74227673	72.75690959	72.75690958
		.30	−2.972291199	70.64798983	70.70298365	70.70298364
	$.25\times10^{-4}$	−.10	.7942514983	66.95865705	66.93332759	66.93332760
		0.	0.	66.16440555	66.16440555	66.16440555
		.10	−.8780042975	65.28640125	65.31667349	65.31667348
		.30	−2.972291199	63.19211435	63.30533191	63.30533191
	$.40\times10^{-4}$	−.10	.7942514983	63.19810976	63.16175929	63.16175931
		0.	0.	62.40385826	62.40385826	62.40385826
		.10	−.8780042975	61.52585396	61.56920107	61.56920107
		.30	−2.972291199	59.43156706	59.59318267	59.59318266

Table 1: Computer output from numerical integration of (7) using Maple.

the mean age of death. Call the three time periods x, y, z. Then $\mu(\cdot, x) = \mu^*(\cdot, x)$ and $\mu(\cdot, z) = \mu^*(\cdot, z)$ and $\mu(\cdot, x) > \mu(\cdot, z)$. By the intermediate value theorem of calculus, during the middle period $\mu^*(\cdot, y)$ progresses through all values between $\mu^*(\cdot, x)$ and $\mu^*(\cdot, z)$. Because of the equivalence of the starred and unstarred forms in time periods x and z, it stands to reason that like the starred force of mortality, $\mu(\cdot, y)$ also attains all values between $\mu(\cdot, x)$ and $\mu(\cdot, z)$. Why, then, is it not observable?

I asked earlier, *which* life expectancy is biased? If the force of mortality changes then I agree with Bongaarts and Feeney that $e_0(t)$ from the *status quo ante* is not a good estimator, but life expectancy from current rates, what I call $e_0^\dagger(t)$, will get the job done. And if $e_0^\dagger(t)$ is what Bongaarts and Feeney mean by life expectancy, then it is in even less need of a correction factor than $e_0(t)$. In the above scenario, $e_0(x)$ plus Bongaarts and Feeney's correction term, $b^{-1}\log(1 - \frac{d}{dt}e_0^*(y))$, is an approximation whereas $e_0(y)$—which can be calculated whenever $\mu(\cdot, y)$ is observed—is exact.

How does $\frac{d}{dt}e_0^*(t) \neq 0$ render $\mu(\cdot, t)$ unobservable? Put another way, how can $\mu^*(\cdot, t)$ affect the mean age of death without also affecting observed death rates? Bongaarts and Feeney do not justify sufficiently why the force of mortality affecting the mean age of death does not equal the observed force of mortality (and must therefore be estimated in the way they propose). In any case, for methodological purposes, $\mu^*(\cdot, t)$ may be simulated even if it is not 'observed'. If a Gompertz is used, (7) may be calculated directly, as in Table 1.

To summarize the numerical results: For a good variety of values of b, $\mu(0, t)$, and $\frac{d}{dt}e_0^*(t)$, the mean age of death as calculated by the first moment equals life expectancy as calculated with $\mu^*(\cdot, t)$. Ceteris paribus, Bongaarts and Feeney's formula for $e_0^*(t)$ puts on a poor show. At least when assuming a Gompertz schedule, whenever mortality rates are observable, life expectancy calculated in the usual way is the best unbiased estimator of the mean age of death.

Conclusion

Bongaarts and Feeney propose a corrected life expectancy, $e_0^*(t)$, that is not self-consistent. When the force of mortality is assumed to be $\mu^*(\cdot, t)$, ostensibly corresponding to $e_0^*(t)$, the mean age at death (first moment) does not equal the proposed $e_0^*(t)$. On the other hand, garden-variety life expectancy comes out equal to the moment-based mean age, provided the correct force of mortality is used. This is demonstrated numerically.

This point hinges on the observability of the force of mortality (operationalized, of course, as observed age-specific death rates, i.e. ${}_nM_x$ values). I question why the observed force of mortality should be different from that affecting the mean age at death when the rate of change of $e_0^*(t)$ is nonzero. The approach of Bongaarts and Feeney is interesting, but it would be premature to dismiss previously-held substantive notions of mortality change based on their proposals. I wholeheartedly agree with Bongaarts and Feeney, however, that we ought neither regard mortality measurement as a closed subject, nor accept principles underlying methods without question.

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