Soc 221B "Cheat sheet" on logs and exponentials

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(Note that "$\equiv$" means "is defined as", or "is exactly the same as",
whereas "$=$" means "is equal to".)

1 Basic definitions

$$\exp(x) \equiv e^x$$

(where $e \doteq 2.71828\ldots$)

The logarithm is the inverse of $\exp(\cdot)$:

$$\log\left(\exp(x)\right) \equiv x$$

and the exponential is the inverse of $\log(\cdot)$:

$$\exp\left(\log(x)\right) \equiv x$$

2 Manipulation rules

$$\exp(a+b) = \exp(a) \times \exp(b)$$
$$\exp(a-b) = \exp(a) \div \exp(b)$$
$$\exp(ab) = [\exp(a)]^b$$

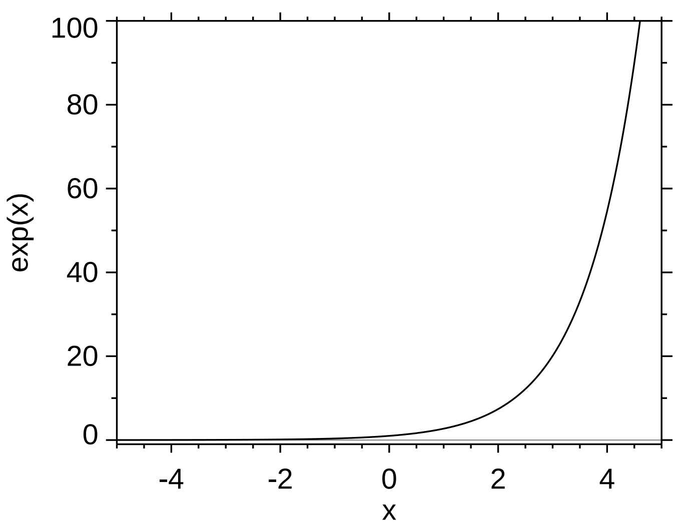
$$\log(a \times b) = \log(a) + \log(b)$$
$$\log(a \div b) = \log(a) - \log(b)$$
$$\log(a^b) = b \times \log(a)$$
$$\log(a+b) = \log(a+b) \quad \text{(no further manipulation)}$$

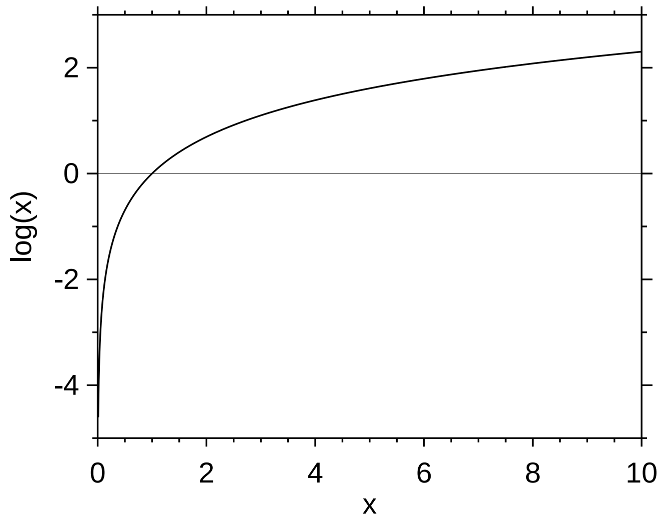
3 Critical values

$$\lim_{x\to-\infty} \exp(x) = 0$$
$$\exp(0) = 1$$
$$\exp(1) = e \doteq 2.71828\ldots$$
$$\lim_{x\to\infty} \exp(x) = \infty$$

$$\lim_{x\downarrow 0} \log(x) = -\infty$$
$$\log(1) = 0$$
$$\log(e) = 1$$
$$\lim_{x\to\infty} \log(x) = \infty$$

4 Graphs





5 [*] More in-depth mathematics

$\log(x)$ is defined for $x > 0$.
$\exp(x)$ is defined $\forall x \in \mathbb{R}$.

If $\log(\cdot)$ and $\exp(\cdot)$ are defined as inverses of each other, isn't that circular reasoning? Yes. There is an alternative definition of the logarithm that provides a way out of the "chicken and egg" problem:

$$\log(x) = \int_1^x \frac{1}{t} dt$$

and note also that:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

(a full treatment of the intricacies here is way beyond the present scope; see any good calculus textbook).

Also n.b., $\log[F(\cdot)]$ is a monotone transformation:

$$\arg\max F(\cdot) = \arg\max \log[F(\cdot)]$$

6 [*] The number e

The "special number" $e \doteq 2.7183$ pops up in a wide variety of places.

It is intimately related to the concept of percentage change.

Suppose a population is growing at some growth rate, say 2% per annum. How long will it take this population to double? The exponential gives us the answer:

$$P_{(\text{future})} = P\exp(r\Delta_t)$$

where P is the current population, r is the growth rate ($2\% = .02$ as we have stipulated) and Δ_t denotes how many units of time into the future we wish to go. Since the growth rate is per annum, the units of Δ_t must be in years. I prefer the $\exp(x)$ notation over the e^x notation because I find this harder to read:

$$P_{(\text{future})} = Pe^{(r\Delta_t)}$$

but the two equations are the same.

Now, if we want to know how long it will take for the population to double, then $P_{(\text{future})} = 2 \times P = 2P$. Substitute:

$$2P = P\exp(r\Delta_t)$$

then:

$$2 = \exp(r\Delta_t)$$
$$\log(2) = r\Delta_t$$

So for doubling:

$$\Delta_t = \log(2)/r.$$

In this specific case:

$$\Delta_t \doteq 0.693147/0.02 \doteq 34.657$$

or about 34.66 years.

You may have heard the rule of thumb that 70 divided by the growth rate (in percent) is the doubling time. Then:

$$70 \div 2[\%] = 35 \text{ years};$$

the rule of thumb is not too far off, and indeed it *comes from* the fact that $\log(2) \approx 0.70$ (the decimals cancel — we use 70 rather than 0.70 but 2[%] instead of 0.02).

(If you require convincing, make a spreadsheet, e.g. in Excel, where you start with "100" in a cell, and enter a formula in the cell below: $1.02\times$ the cell above. Then copy the formula down the column. How many cells down do you have to go to double [i.e. get to 200]? Here you have percentage growth with no recourse to e. There is a discrete time *vs.* continuous time disconnect here, so things will not line-up perfectly, but it's very close.)

These doubling time calculations link percentage growth to $\exp(\cdot)$ and hence to the number e.

How does this work out? This is a bit like asking why the ratio of the circumference to the diameter of a circle is always $\pi \doteq 3.14159$. Exponential growth *is* percentage growth. In applied mathematics, there is no getting away from the number e.