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Models for estimating empirical Gompertz mortality: With an application to evolution of the Gompertzian slope

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Abstract

Using data from the Human Mortality Database (HMD), and five different modeling approaches, we estimate Gompertz mortality parameters for 7,704 life tables. To gauge model fit, we predict life expectancy at age 40 from these parameters, and compare predicted to empirical values. Across a diversity of human populations, and both sexes, the overall best way to estimate Gompertz parameters is weighted least squares, although Poisson regression performs better in 996 cases for males and 1,027 cases for females, out of 3,852 populations per sex. We recommend against using unweighted least squares unless death counts (to use as weights or to allow Poisson estimation) are unavailable. We also recommend fitting to logged death rates. Over time in human populations, the Gompertz slope parameter has increased, indicating a more severe increase in mortality rates as age goes up. However, it is well-known that the two parameters of the Gompertz model are very tightly (and negatively) correlated. When the slope goes up, the level goes down, and, overall, mortality rates are decreasing over time. An analysis of Gompertz parameters for all of the HMD countries shows a distinct pattern for males in the formerly socialist economies of Europe.

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1 Introduction

We examine methods for estimating the Gompertz (1825) mortality relationship in human populations, and describe the long-term evolution of its slope parameter. The Gompertz parametric description of mortality at older ages, often called a law (e.g., Brillinger 1961, Carnes et al. 1996, Le Bras 2008), is one of the oldest mathematical formulations in demography (Turner and Hanley, 2010). It has been applied widely, not only in actuarial science and human demography (e.g., Greenwood and Irwin 1939, Bowers et al. 1997) but in biology (Mueller et al. 1995, Kirkwood 2015) as well as in medicine, particularly oncology (e.g., Sacher 1956, Neafsey and Lowrie 1993).

Our primary goal is to determine the best way to estimate the Gompertz model for mortality rates above age 40 in human populations. A second objective is to document the evolution of the Gompertz slope parameter over time, for a wide variety of human populations. This work is a type of meta-analysis using big (relatively speaking) databases of populations; we analyze 3,852 populations per sex, from 41 polities. We find that for ecological analysis of older population mortality, weighted least squares is the best-performing model on average, although there are instances in which Poisson regression produces better-fitting models. Over time, as human populations have achieved greater longevity, the Gompertzian slope (β) has steadily increased. This indicates a more—not less—severe relationship between age and rising mortality above age 40. However, the baseline mortality rate at age 40 has declined, making up for the concomitant increases in β. This underscores the multidimensionality of population average longevity. That is to say, β must be interpreted with care. This paper begins with a description of the data and several techniques to estimate the Gompertz model, followed by a comparison of models to data, and then a discussion of the long-term changes.

1.1 Background

In words, the Gompertz mortality model is that the force of mortality (μ_x) increases exponentially with age (above some threshold age, usually taken to be somewhere between 35 and 45). As an equation, it is: $M_x \approx \alpha \exp(\beta x)$, where x is age and α and β are free parameters (i.e., they may vary between populations and over time); there are other, equivalent, forms of

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the relationship (Missov et al., 2015). Gompertzian mortality is often applied directly to death rates (M_x), as we do here. When μ_x follows the Gompertz relationship, so does M_x, because μ_x is the instantaneous form of m_x, the life table death rate (Keyfitz, 1985, p.36), and M_x is used as a drop-in replacement for m_x when estimating life tables from real-world data (Wachter, 2014, p.154).

Although the Gompertz mortality model is widely used, there is no consensus on the best way to estimate its parameters (i.e., α, β) from observational data. Wachter (2014, p.69) recommends taking the logarithm of M_x, which converts the exponential on the right hand side into a sum of two components, easily estimated as a linear regression; cf. also Horiuchi and Coale (1982). Preston et al. (2001, p.193) suggests a technique using three consecutive life table survivorship (ℓ_x) values; Namboodiri (1991, p.95) presents a similar approach. Earlier work on this subject includes Trachtenberg (1924), Stoner (1941), Brennan (1949), and Sherman and Morrison (1950).

We evaluate five ways to estimate the Gompertz model, described below. All these may be estimated using common statistical software packages, without extensive programming. These approaches use data on mortality at all ages above 40 and below 100. The Gompertz model is not a good description of mortality among centenarians (Horiuchi and Coale 1990, Horiuchi and Wilmoth 1998). Our five approaches use linear or nonlinear regression, with or without weights, and Poisson regression. The latter method was introduced in this context by Abdullatif and Noymer (2016), building on an approach suggested by Brillinger (1986). Garg et al. (1970) and Prentice and Shaarawi (1973) suggest an alternate maximum likelihood method.

2 Materials and methods

2.1 Data

We analyze every population in the Human Mortality Database as of the time of this writing (Barbieri et al. 2015, Human Mortality Database 2017). By "population", we mean permutations of country×year×sex, of which there are 7,704 in total. The countries included in the Human Mortality Database (HMD) are listed in table 1, along with their start and end dates.

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Table 1: Included populations with dates and number of years.

HMD abbreviation	Country	start year	end year	N years
AUS	Australia	1921	2014	94
AUT	Austria	1947	2014	68
BEL	Belgium	1841	2015	170
BGR	Bulgaria	1947	2010	64
BLR	Belarus	1959	2014	56
CAN	Canada	1921	2011	91
CHE	Switzerland	1876	2014	139
CHL	Chile	1992	2005	14
CZE	Czech Rep.	1950	2014	65
DEUTE	Germany (E.)	1956	2013	58
DEUTW	Germany (W.)	1956	2013	58
DNK	Denmark	1835	2014	180
ESP	Spain	1908	2014	107
EST	Estonia	1959	2013	55
FIN	Finland	1878	2015	138
FRATNP	France	1816	2014	199
GBRTENW	England & Wales	1841	2013	173
GBR-NIR	N. Ireland	1922	2013	92
GBR-SCO	Scotland	1855	2013	159
GRC	Greece	1981	2013	33
HUN	Hungary	1950	2014	65
IRL	Ireland	1950	2014	65
ISL	Iceland	1838	2013	176
ISR	Israel	1983	2014	32
ITA	Italy	1872	2012	141
JPN	Japan	1947	2014	68
LTU	Lithuania	1959	2013	55
LUX	Luxembourg	1960	2014	55
LVA	Latvia	1959	2013	55
NLD	Netherlands	1850	2012	163
NOR	Norway	1846	2014	169
NZL-NP	New Zealand	1948	2013	66
POL	Poland	1958	2014	57
PRT	Portugal	1940	2012	73
RUS	Russia	1959	2014	56
SVK	Slovakia	1950	2014	65
SVN	Slovenia	1983	2014	32
SWE	Sweden	1751	2014	264
TWN	Taiwan	1970	2014	45
UKR	Ukraine	1959	2013	55
USA	United States	1933	2014	82

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2.2 Methods

We estimate five models of over-40 mortality, as follows:

$$\text{OLS:} \quad \log(M_x) = \alpha + \beta x \tag{1}$$
$$\text{WOLS:} \quad \log(M_x) = \alpha + \beta x \tag{2}$$
$$\text{Poisson:} \quad \log(D_x) = \alpha + \beta x + \log(K_x) \tag{3}$$
$$\text{NLLS:} \quad M_x = \alpha \exp(\beta x) \tag{4}$$
$$\text{WNLLS:} \quad M_x = \alpha \exp(\beta x), \tag{5}$$

where D_x is the number of deaths in a five-year age group from age x to age $x + 4$, inclusive ($x = 40, 45, \ldots, 95$), K_x is the person-years at risk (exposure) in the same age group, and $M_x = D_x / K_x$ is the mortality rate in the age group. The M_x values were calculated from the deaths and exposures; HMD pre-calculated M_x were not used. Using five-year age groups smooths heaping caused by digit preference (Shryock et al., 1971, p.439). Model (1) is estimated by ordinary least squares (OLS); model (2) by weighted ordinary least squares (WOLS) with D_x as weights; model (3) by Poisson regression (using maximum likelihood); model (4) by nonlinear least squares (NLLS); and model (5) by weighted nonlinear least squares (WNLLS) with D_x as weights. The use of deaths as weights brings the least squares estimates closer to those of the maximum likelihood approach (Carey et al., 1993), since the likelihoods are calculated with the cell counts (i.e., D_x). We also test least squares without weights (i.e., comparing the fit with and without weights is one of our goals). We used Stata v.13.1 software (StataCorp LLC, College Station, Texas, USA).

All of (1)–(5) are models of the same basic relationship, Gompertzian mortality above age 40. These are two-parameter models, the estimated coefficients of which are named α and β. However, the parameter estimates are not interchangeable across models. The input data are on different scales: $\log(M_x)$ for model (1), weighted $\log(M_x)$ for model (2), deaths (with exposure as an offset, see Agresti, 2002, p.385) for model (3), M_x for model (4), and weighted M_x for model (5).

Figure 1 illustrates the data and fitting techniques, using Japanese males, 2014, as an example. The upper left quadrant, labeled "OLS", shows the data with a logarithmic vertical axis, and the all the points are drawn the same size to signify that they have the same weight in the estimation. The parameters in (1) are the slope and intercept of the line in the figure. The

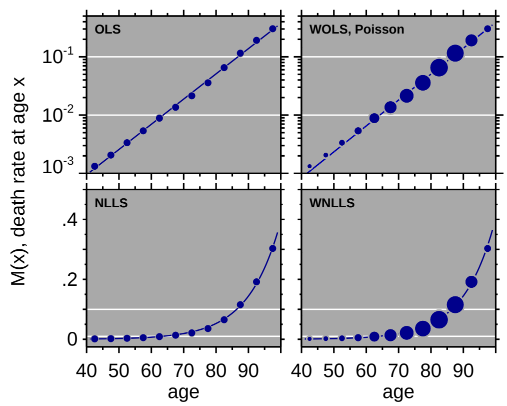

Figure 1: Graphical illustration of five modeling approaches, using Japanese males in 2014 as an example.

upper right quadrant, labeled "WOLS, Poisson", also has a logarithmic vertical axis, but the area of each ploting symbol is proportional to the number of deaths (D_x) that went into the rate calculation for that point. Deaths in the first several age groups are comparatively few because death rates are relatively low at these ages, while in the 95–99 age group there are few deaths despite the rate being high, because the population at risk is small. This conveys the approach of the weighted OLS (WOLS) estimator. The Poisson rate regression is conceptually similar, but log deaths, not rates, are modeled, with population as an offset. The parameters in (2) or (3) are the slope and intercept of the line in the figure; at the scale of figure 1, there is no discernible difference between the coefficients from WOLS or Poisson.

The bottom left quadrant of figure 1, labeled "NLLS", shows the data with a linear vertical axis, with all the points the same size. The two horizontal gridlines are at the same values as those in the top row of the figure, but appear closer together because of the linear scale of the vertical axis. The nonlinear least squares parameters from (4) are used to draw the curve shown. The bottom right quadrant ("WNLLS") is similar to the bottom left, but weights have been used, and, again, the size of the plotting symbols reflects the weights. The coefficients from (5) provide the curve drawn.

Judgment of model fit against the input data using the coefficient of determination (R^2) is tricky, because fitting weighted and unweighted quantities makes fair comparison difficult (Willett and Singer 1988, Scott and

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Wild 1991). The same applies to logged vs. direct scale, i.e., models (1)–(3) vs. (4)–(5). Other approaches to model selection such as BIC or AIC are also problematic in this context, for the same reason: the input data are not the same across the models. Also, not all these models produce likelihood statistics, although there are maximum likelihood estimators for linear regression (e.g. Freedman, 2009, p.271). There is a large literature on model selection in complex contexts—for example, Varin and Vidoni (2005); Leeb and Pötscher (2005); Ng and Joe (2014); Leeb and Pötscher (2017); Fithian et al. (2017)—which is potentially applicable here. For the present purpose, we judge the fit of the models to data using what we call a functional approach: we use the estimates to calculate e_{40} (life expectancy at age 40). Expected (i.e., model predicted, using the estimated coefficients) minus observed (i.e., empirical) e_{40}, squared, then provides a measure of fit. This is an appropriate way to assess model fit not only because it translates the results into a common metric, but since it is a life table (i.e., demographic) statistic, independent of the various estimators in (1)–(5). Life expectancy at age 40 has the desirable property that it takes into account data from all ages 40 and above.

Calculating life expectancy at age 40 involves several steps. Model-predicted $\widehat{M}_x$ values are generated. For OLS, WOLS, and Poisson, the prediction equation is $\widehat{M}_x = \exp(\hat{\alpha} + \hat{\beta}x)$, while for NLLS and WNLLS, it is $\widehat{M}_x = \hat{\alpha}\exp(\hat{\beta}x)$. The prediction equations obviously resemble the estimation equations, (1)–(5), but are not exactly the same. From these predicted $\widehat{M}_x$, we generated the life table probability of dying, q_x (Wachter 2014):

$$q_x = \widehat{M}_x \Big/ \left(1 + \widehat{M}_x/2\right), \text{ for } x = 40, 41, \ldots \omega, \tag{6}$$

where ω is the hypothetical age beyond which nobody lives (we used $\omega = 110$, which works fine in practice). The next step is to calculate the survivor column, ℓ_x:

$$\ell_x = \exp\left(\sum_{a=40}^{x} \log\left(1 - q_a\right)\right) \text{ for } x = 41, \ldots, \omega, \tag{7}$$

with $\ell_{40} \equiv 1$. Lastly, we calculate e_{40} as the integral of the ℓ_x (a sum in discrete approximation):

$$e_{40} = \sum_{x=40}^{\omega} \ell_{x+1} + a_x \cdot q_x \cdot \ell_x, \text{ where:} \tag{8}$$
$$a_x = \begin{cases} 1/\widehat{M}_x & \text{if } x = \omega \\ 0.5 & \text{otherwise.} \end{cases}$$

This completes the calculation of the modeled e_{40}, which is then compared to the empirical e_{40} taken from the HMD. There is also a formula from Gompertzian μ_x directly to ℓ_x (Pollard, 1972, p.17), but we prefer our approach since our parameters refer to a Gompertizan relationship in the M_x, not μ_x. The modeled e_{40} is based entirely upon two parameters, the estimated $\hat{\alpha}, \hat{\beta}$ from (1)–(5) (as applicable). The prediction equations (as applicable), and the chain of calculations in (6)–(8) do not add new information. In comparison, the empirical e_{40} statistic is drawn from the HMD life table by single year of age. The empirical e_{40} is a summary statistic not of a two-parameter mortality model but of the death rates by single year of age above 40. For each sex×country combination, containing N_{years} observations, we calculated the root mean square error for each model:

$$\text{RMSE} = \sqrt{\frac{1}{N_{\text{years}}} \sum_{\text{years}} \left(e_{40}^{\text{model}} - e_{40}^{\text{HMD}}\right)^2}, \tag{9}$$

which we use as a yardstick to rank the model approaches on a per-country and per-sex basis. The lower the RMSE, the better fitting is the the model. The e_{40} and RMS calculations were performed using IDL v.8.6 (Exelis Visual Information Solutions, Inc., Broomfield, Colorado, USA).

3 Results

3.1 Root mean squared error of the models

Table 2 presents the RMS error, as calculated in (9), for each of the five models, on a per-country and per-sex basis, in units of years of e_{40}. The first line of the table is for all 3,852 life tables (per sex), pooled. Note that each subsequent line of the table is a country-specific summary for a variable number of years (given in table 1). A few things are clear. First, on a

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Table 2: RMS error of models for life expectancy at age 40, nested in country × sex combinations.

	Root Mean Square (RMS) error for life expectancy at age 40 (e_{40}) fit (years)									
	Males					Females				
Country	OLS	WOLS	Poisson	NLLS	WNLLS	OLS	WOLS	Poisson	NLLS	WNLLS
All countries	0.294	0.072*	0.088†	3.179	1.345	0.258	0.073*	0.096†	2.787	1.593
Australia	0.390	0.035*	0.049†	2.379	1.243	0.154	0.058†	0.057*	1.450	0.856
Austria	0.238	0.076†	0.073*	2.486	1.044	0.243	0.057*	0.060†	2.689	1.537
Belgium	0.265	0.058*	0.079†	2.643	0.944	0.243	0.044*	0.076†	2.340	1.047
Bulgaria	0.493	0.054*	0.068†	4.283	2.407	0.577	0.055*	0.074†	5.246	3.577
Belarus	0.131	0.055*	0.063†	1.985	0.866	0.268	0.085*	0.104†	3.077	2.560
Canada	0.277	0.035*	0.047†	2.204	1.032	0.130	0.041*	0.048†	1.529	0.842
Switzerland	0.216	0.038*	0.054†	3.089	0.799	0.200	0.057†	0.055*	2.762	1.198
Chile	0.251	0.063*	0.073†	1.528	0.985	0.087	0.050*	0.058†	0.931	0.644
Czech Rep.	0.372	0.070†	0.058*	2.749	1.239	0.210	0.037*	0.044†	3.224	1.871
Germany (E.)	0.273	0.126*	0.131†	2.665	1.435	0.180	0.062*	0.095†	2.936	1.869
Germany (W.)	0.307	0.089†	0.085*	2.254	1.317	0.241	0.046*	0.082†	2.303	1.418
Denmark	0.237	0.041*	0.062†	3.182	0.995	0.232	0.064*	0.078†	2.412	0.942
Spain	0.239	0.111*	0.150†	2.834	1.580	0.336	0.141*	0.223†	2.704	2.434
Estonia	0.122	0.041*	0.046†	2.762	1.008	0.263	0.060*	0.113†	2.744	1.786
Finland	0.229	0.053*	0.074†	4.141	1.412	0.285	0.066*	0.068†	3.335	1.395
France	0.232	0.051*	0.095†	2.418	0.797	0.275	0.062*	0.092†	2.209	0.876
England & Wales	0.462	0.081†	0.065*	2.769	1.571	0.219	0.036*	0.045†	2.200	1.514
N. Ireland	0.332	0.077†	0.076*	2.602	1.054	0.142	0.037*	0.053†	2.023	0.911
Scotland	0.369	0.070†	0.064*	2.107	1.131	0.138	0.042†	0.040*	1.765	0.932
Greece	0.292	0.049*	0.066†	2.173	1.452	0.293	0.046*	0.068†	3.445	3.101
Hungary	0.205	0.057†	0.039*	2.246	1.047	0.252	0.032*	0.072†	2.581	1.376
Ireland	0.497	0.083†	0.059*	2.513	1.872	0.192	0.043†	0.022*	1.928	1.682
Iceland	0.357	0.189*	0.210†	8.279	3.226	0.377	0.178*	0.197†	5.896	2.898
Israel	0.381	0.050†	0.043*	1.687	1.349	0.315	0.046†	0.031*	1.537	1.290
Italy	0.230	0.078*	0.115†	2.712	1.028	0.169	0.100*	0.140†	3.194	1.730
Japan	0.208	0.056*	0.069†	2.639	1.050	0.153	0.046*	0.123†	2.100	1.101
Lithuania	0.144	0.074*	0.090†	2.459	1.344	0.407	0.072*	0.147†	3.128	2.579
Luxembourg	0.409	0.087†	0.071*	3.741	1.453	0.294	0.068*	0.080†	3.108	1.619
Latvia	0.138	0.049*	0.061†	2.599	1.495	0.299	0.077*	0.137†	2.738	1.789
Netherlands	0.310	0.038*	0.058†	2.647	1.113	0.237	0.062*	0.070†	2.322	1.010
Norway	0.332	0.056*	0.084†	1.924	1.124	0.352	0.088†	0.086*	1.689	1.195
New Zealand	0.481	0.033*	0.037†	2.517	1.555	0.199	0.063†	0.054*	1.614	0.981
Poland	0.235	0.037*	0.040†	2.770	1.040	0.214	0.048*	0.069†	2.922	1.596
Portugal	0.189	0.035*	0.067†	2.775	1.228	0.177	0.033*	0.077†	2.741	2.241
Russai	0.153	0.047*	0.054†	2.594	1.018	0.291	0.058*	0.096†	3.342	2.154
Slovakia	0.161	0.029*	0.033†	2.330	0.623	0.156	0.041*	0.051†	2.930	1.618
Slovenia	0.154	0.045*	0.049†	2.218	0.659	0.276	0.037*	0.081†	1.729	1.074
Sweden	0.242	0.043*	0.089†	2.669	1.083	0.264	0.060*	0.089†	2.310	1.000
Taiwan	0.344	0.122*	0.143†	2.778	0.994	0.233	0.065*	0.084†	2.349	1.346
Ukraine	0.129	0.057*	0.065†	2.181	1.034	0.269	0.058*	0.091†	3.198	2.222
United States	0.267	0.029*	0.031†	2.112	0.841	0.201	0.062*	0.066†	1.913	0.816

* best model, † second-best model. OLS is third-best model, WNLLS is fourth-best (and NLLS, last) for all populations.

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country-nested basis, the best model is always either WOLS (31 times for males, 34 times for females) or Poisson (10 times for males, 7 times for females). These two models also have very similar RMS values (e.g., for males overall, 0.072 for WOLS vs. 0.088 for Poisson). Third place always belongs to OLS estimation (viz., in all countries and both sexes), but the RMS error for OLS was about one power of ten higher than for WOLS/Poisson (e.g., 0.294 for males overall). In fourth place is always weighted nonlinear least squares (WNLLS), but its RMS error was on the order of another power of ten higher than that of OLS (1.345 for males overall). Unweighted nonlinear least squares (NLLS) was always in last place, with approximately double the RMS error of WNLLS.

Overall, following (9) (and table 2), Poisson was the best-fitting model in 17 populations (i.e., life tables nested into countries) out of 82 (2 sexes×41 countries), or about 21% of the time. By conventional criteria, we would not say that that Poisson is better only on "chance" occasions, since these exceed 5%. However, there appears to be no rhyme or reason for which populations show a better fit for Poisson than WOLS in table 2. There is little consistency across the sexes; in only three national populations is Poisson the best fit for both sexes (Scotland, Ireland, and Israel). Without the hierarchical nesting of populations into countries (i.e., considering all 3,852 sex-specific life tables simultaneously), Poisson appears somewhat more viable, being the better fit 996 times for males (26% of the time), and 1027 (27%) for females. As noted, the WOLS and Poisson models are closer to each other in RMS error than either one is to the next-best model, unweighted OLS. Again considering all 3,852 results without country nesting, when WOLS is a better fit than Poisson, the RMS difference is 0.056 years for males, while when Poisson is the better model, the RMS difference is 0.061 years. The female models were closer together: for the 3,852 female results, when WOLS is the better model its RMS error is 0.003 years better than the corresponding Poisson model, while when Poisson is better, it is by 0.002 years. In short, WOLS has more flattering RMS error statistics, but the Poisson model is viable.

Similar to the Poisson models, we also estimated negative binomial (NB) regressions. This is the same as (3) but with a different likelihood function (Hilbe, 2011). These models failed to converge for at least one year×sex combination in three countries (Iceland, Luxembourg, and Northern Ireland). From a practical perspective, nonconvergence is a major obstacle. For this reason, we do not consider NB to be a promising alternative to the models we present in full. There is only one national population (Taiwan, both sexes) in which the NB model performed the best in the e_{40} comparison out of all the models. Moreover, in only one other population (Lithuania males) did NB outperform the other count model (i.e., Poisson), for second place. Given that the pool of populations in which we were able to estimate both of the count models is 39 polities×2 sexes, NB just doesn't seem to do as well as WOLS or Poisson for estimating Gompertzian mortality, except for some idiosyncratic cases.

Using predicted versus empirical e_{40} as the measuring stick, so to speak, reflects a choice. Because Gompertz models can be used in model life table construction, and because life expectancy calculation is one of the major uses of life tables, we believe e_{40} is a reasonable choice. Nonetheless, it is not the only possible measuring stick. The Gompertz model is a fit of M_x data, so the e_{40} RMS is not a goodness-of-fit statistic in the strict sense (i.e., the Gompertz coefficients are not fit from e_{40}). To address this, table 3 gives mean coefficients of determination (R^2), using the same nesting as table 2. The R^2 statistics are a measure of each model's fit to the underlying data.

Caution is warranted comparing across columns of table 3, because the models are not all fitting the same quantity; some are fitting $\log(M_x)$ and some M_x; some are weighed (see Rivadeneira and Noymer 2017, p.45, for this R^2 formula, and cf. also Kvålseth 1985). Consider the WOLS and Poisson columns; these models share the same prediction equation and R^2 formula, but have different estimators. Least squares is "BLUE"; it is the best linear unbiased estimator. Thus WOLS always beats Poisson when measured by the R^2 (see table 3), although they are often tied when rounded to four significant digits. While the coefficients of determination have the shortcoming that they are not directly comparable between model type, they are a direct measure of model-to-data, so are relevant to the overall exercise.

At the suggestion of a reviewer, table 4 considers the same relationship as table 2, but for life expectancy at age 50. We re-estimated all the relationships, using data on ages 50–99 instead of 40–99, and we used the resulting Gompertz coefficients to calculate e_{50}, which was compared to the HMD values. Qualitatively, the results are much the same as for table 4; specifically, the recommended model is still WOLS. Nonetheless, there are some differences. Poisson models outperform WOLS in fewer instances: only 537 for males and 613 for females, and, when nested into countries, only 3 times for males and twice for females. Also, the RMS errors are

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Table 3: Coefficient of determination (model R^2), nested in country × sex combinations.

	Coefficient of determination (R^2) for fitting M_x									
	Males					Females				
Country	OLS	WOLS	Poisson	NLLS	WNLLS	OLS	WOLS	Poisson	NLLS	WNLLS
All countries	0.9875	0.9927	0.9927	0.9800	0.9891	0.9775	0.9907	0.9907	0.9621	0.9885
Australia	0.9962	0.9969	0.9969	0.9948	0.9945	0.9818	0.9959	0.9959	0.9650	0.9953
Austria	0.9976	0.9975	0.9975	0.9940	0.9956	0.9751	0.9941	0.9941	0.9545	0.9918
Belgium	0.9951	0.9938	0.9937	0.9894	0.9940	0.9841	0.9891	0.9890	0.9782	0.9929
Bulgaria	0.9927	0.9955	0.9955	0.9802	0.9810	0.9709	0.9920	0.9920	0.9471	0.9711
Belarus	0.9973	0.9973	0.9973	0.9914	0.9915	0.9718	0.9944	0.9944	0.9520	0.9832
Canada	0.9976	0.9980	0.9980	0.9953	0.9949	0.9821	0.9969	0.9969	0.9653	0.9960
Switzerland	0.9964	0.9968	0.9967	0.9893	0.9942	0.9848	0.9938	0.9938	0.9687	0.9924
Chile	0.9985	0.9983	0.9983	0.9961	0.9946	0.9019	0.9989	0.9989	0.8336	0.9971
Czech Rep.	0.9964	0.9971	0.9971	0.9949	0.9958	0.9774	0.9973	0.9973	0.9583	0.9916
Germany (E.)	0.9972	0.9971	0.9971	0.9955	0.9956	0.9716	0.9950	0.9950	0.9480	0.9914
Germany (W.)	0.9978	0.9977	0.9977	0.9974	0.9969	0.9712	0.9946	0.9946	0.9478	0.9933
Denmark	0.9944	0.9947	0.9947	0.9834	0.9918	0.9854	0.9908	0.9907	0.9766	0.9921
Spain	0.9946	0.9938	0.9938	0.9920	0.9858	0.9763	0.9868	0.9867	0.9669	0.9832
Estonia	0.9961	0.9967	0.9967	0.9848	0.9900	0.9726	0.9939	0.9939	0.9462	0.9903
Finland	0.9836	0.9923	0.9923	0.9588	0.9882	0.9903	0.9889	0.9888	0.9825	0.9898
France	0.9925	0.9886	0.9885	0.9895	0.9892	0.9835	0.9877	0.9876	0.9786	0.9896
England & Wales	0.9944	0.9943	0.9943	0.9902	0.9895	0.9820	0.9944	0.9944	0.9636	0.9901
N. Ireland	0.9939	0.9935	0.9934	0.9864	0.9917	0.9865	0.9933	0.9933	0.9760	0.9922
Scotland	0.9949	0.9929	0.9929	0.9964	0.9954	0.9875	0.9932	0.9932	0.9805	0.9952
Greece	0.9978	0.9975	0.9975	0.9960	0.9945	0.9520	0.9909	0.9909	0.9223	0.9840
Hungary	0.9971	0.9972	0.9972	0.9954	0.9961	0.9752	0.9952	0.9952	0.9639	0.9929
Ireland	0.9959	0.9960	0.9960	0.9960	0.9937	0.9777	0.9968	0.9968	0.9610	0.9938
Iceland	0.8451	0.9585	0.9577	0.7901	0.9255	0.9318	0.9682	0.9677	0.8977	0.9424
Israel	0.9973	0.9972	0.9972	0.9960	0.9937	0.9563	0.9966	0.9966	0.8789	0.9937
Italy	0.9941	0.9918	0.9917	0.9922	0.9908	0.9811	0.9885	0.9884	0.9697	0.9858
Japan	0.9978	0.9979	0.9979	0.9954	0.9952	0.9723	0.9925	0.9924	0.9512	0.9943
Lithuania	0.9959	0.9953	0.9953	0.9893	0.9887	0.9674	0.9899	0.9899	0.9185	0.9818
Luxembourg	0.9910	0.9938	0.9938	0.9672	0.9792	0.9725	0.9907	0.9907	0.9388	0.9837
Latvia	0.9961	0.9960	0.9960	0.9907	0.9923	0.9707	0.9935	0.9935	0.9496	0.9890
Netherlands	0.9955	0.9946	0.9946	0.9905	0.9947	0.9839	0.9889	0.9888	0.9753	0.9937
Norway	0.9928	0.9905	0.9904	0.9942	0.9936	0.9818	0.9869	0.9868	0.9790	0.9938
New Zealand	0.9954	0.9961	0.9961	0.9932	0.9930	0.9769	0.9966	0.9966	0.9508	0.9943
Poland	0.9973	0.9985	0.9985	0.9921	0.9943	0.9722	0.9960	0.9960	0.9510	0.9904
Portugal	0.9958	0.9954	0.9954	0.9925	0.9911	0.9741	0.9920	0.9919	0.9612	0.9900
Russai	0.9961	0.9976	0.9976	0.9877	0.9904	0.9696	0.9941	0.9941	0.9479	0.9830
Slovakia	0.9979	0.9982	0.9982	0.9928	0.9948	0.9786	0.9971	0.9971	0.9688	0.9916
Slovenia	0.9976	0.9973	0.9973	0.9912	0.9938	0.9569	0.9928	0.9927	0.9135	0.9932
Sweden	0.9877	0.9887	0.9886	0.9818	0.9915	0.9900	0.9864	0.9863	0.9881	0.9910
Taiwan	0.9941	0.9954	0.9954	0.9860	0.9887	0.9666	0.9972	0.9972	0.9428	0.9927
Ukraine	0.9975	0.9977	0.9977	0.9931	0.9929	0.9703	0.9951	0.9951	0.9521	0.9855
United States	0.9970	0.9977	0.9977	0.9929	0.9957	0.9799	0.9969	0.9969	0.9613	0.9947

Table 4: RMS error of models for life expectancy at age 50, nested in country × sex combinations.

Country	Root Mean Square (RMS) error for life expectancy at age 50 (e_{50}) fit (years)									
	Males					Females				
	OLS	WOLS	Poisson	NLLS	WNLLS	OLS	WOLS	Poisson	NLLS	WNLLS
All countries	0.171	0.065*	0.078†	2.44	0.927	0.220	0.063*	0.081†	2.30	1.365
Australia	0.203	0.049*	0.056†	1.674	0.787	0.135	0.058*	0.064†	1.250	0.802
Austria	0.151	0.061*	0.064†	1.817	0.727	0.202	0.037*	0.039†	2.304	1.428
Belgium	0.123	0.056*	0.064†	2.024	0.605	0.145	0.042*	0.054†	2.047	1.044
Bulgaria	0.331	0.061*	0.077†	3.025	1.615	0.553	0.061*	0.074†	4.047	2.763
Belarus	0.129	0.062*	0.070†	1.478	0.683	0.306	0.081*	0.095†	2.466	2.050
Canada	0.169	0.053*	0.058†	1.629	0.735	0.141	0.053*	0.062†	1.296	0.764
Switzerland	0.123	0.045*	0.051†	2.285	0.529	0.178	0.041*	0.046†	2.217	1.101
Chile	0.224	0.066*	0.075†	1.130	0.694	0.095	0.060*	0.068†	0.794	0.556
Czech Rep.	0.151	0.050†	0.049*	1.842	0.718	0.229	0.037*	0.039†	2.619	1.562
Germany (E.)	0.168	0.111*	0.118†	1.957	1.015	0.182	0.049*	0.064†	2.505	1.680
Germany (W.)	0.182	0.076*	0.078†	1.607	0.888	0.205	0.033*	0.060†	2.004	1.320
Denmark	0.114	0.047*	0.058†	2.458	0.609	0.127	0.049*	0.064†	1.999	0.876
Spain	0.210	0.114*	0.133†	2.121	1.206	0.422	0.155*	0.208†	2.193	1.972
Estonia	0.103	0.053*	0.060†	2.089	0.750	0.214	0.053*	0.086†	2.304	1.547
Finland	0.119	0.057*	0.070†	3.101	0.849	0.230	0.049*	0.051†	2.738	1.280
France	0.112	0.040*	0.055†	2.069	0.677	0.117	0.042*	0.048†	2.026	1.029
England & Wales	0.224	0.041*	0.043†	1.956	1.022	0.203	0.031*	0.040†	1.706	1.188
N. Ireland	0.137	0.052*	0.059†	1.930	0.696	0.106	0.039*	0.053†	1.691	0.855
Scotland	0.168	0.052*	0.057†	1.594	0.713	0.109	0.042*	0.050†	1.552	0.867
Greece	0.220	0.049*	0.063†	1.684	1.101	0.383	0.038*	0.053†	2.971	2.711
Hungary	0.127	0.042*	0.048†	1.653	0.762	0.214	0.034*	0.051†	2.152	1.216
Ireland	0.264	0.051†	0.043*	1.779	1.254	0.166	0.027†	0.019*	1.547	1.366
Iceland	0.336	0.144*	0.191†	6.722	2.269	0.347	0.132*	0.177†	4.731	2.247
Israel	0.230	0.040†	0.040*	1.288	0.961	0.264	0.041†	0.036*	1.281	1.062
Italy	0.119	0.080*	0.095†	2.180	0.870	0.235	0.105*	0.126†	2.612	1.558
Japan	0.181	0.057*	0.066†	2.002	0.810	0.156	0.056*	0.095†	1.839	1.036
Lithuania	0.143	0.078*	0.094†	1.873	1.007	0.357	0.072*	0.120†	2.558	2.123
Luxembourg	0.199	0.055*	0.063†	2.838	1.019	0.232	0.042*	0.064†	2.559	1.447
Latvia	0.099	0.059*	0.067†	1.972	1.139	0.219	0.063*	0.096†	2.315	1.562
Netherlands	0.176	0.046*	0.053†	2.011	0.762	0.169	0.044*	0.051†	2.007	0.985
Norway	0.126	0.047*	0.062†	1.531	0.695	0.155	0.047*	0.060†	1.428	0.953
New Zealand	0.247	0.034*	0.042†	1.758	0.982	0.152	0.049*	0.054†	1.335	0.838
Poland	0.167	0.051*	0.055†	1.982	0.726	0.244	0.054*	0.066†	2.412	1.372
Portugal	0.142	0.050*	0.064†	2.187	1.046	0.292	0.050*	0.072†	2.347	1.965
Russai	0.109	0.062*	0.069†	1.860	0.694	0.290	0.063*	0.083†	2.714	1.777
Slovakia	0.099	0.038*	0.042†	1.755	0.488	0.203	0.045*	0.050†	2.417	1.388
Slovenia	0.098	0.050*	0.058†	1.656	0.498	0.145	0.045*	0.074†	1.557	1.050
Sweden	0.112	0.048*	0.067†	2.021	0.612	0.131	0.039*	0.052†	1.993	0.984
Taiwan	0.272	0.118*	0.133†	2.055	0.724	0.269	0.085*	0.097†	1.898	1.112
Ukraine	0.119	0.063*	0.069†	1.621	0.715	0.292	0.052*	0.070†	2.610	1.837
United States	0.129	0.045*	0.048†	1.455	0.508	0.145	0.062*	0.068†	1.515	0.671

* best model, † second-best model. OLS is third-best model, WNLLS is fourth-best (and NLLS, last) for all populations.

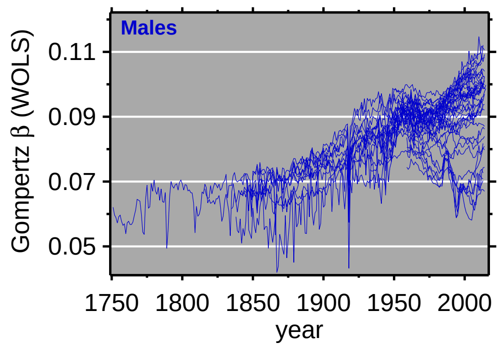

Figure 2: Evolution over time of Gompertzian slope parameter, $\hat{\beta}$, males. Shown are 41 countries, comprising 3,852 annual life tables.

smaller, on average, for predicting life expectancy at age 50, compared to those for e_{40}. As seen in figure 1, the data at ages 40–44 and 45–49 often do not fit as well; it stands to reason that fitting without these points (viz., fitting e_{50}) should improve the RMSE. An interesting extension that would be a natural outgrowth of the table 2/4 comparison (but which is beyond the current scope), would be to quantify how the RMS fit for e_a changes for $10 \leq a \leq 70$.

3.2 The evolution of the Gompertz slope parameter, β

Figures 2 (males) and 3 (females) show the evolution of the Gompertz β estimates over time, for the complete data set. Based on the performance discussed above, we use the WOLS estimates for this analysis. Unlike (to the best of our knowledge) the preceding subsection on comparing fit, the results in this subsection have been explored before—for example figure 1 of Strulik and Vollmer (2013), for the case of Sweden. Several features are noteworthy. Over time, the Gompertz β has gotten larger. This is part of the life table "rectangularization" process that is a well-studied phenomenon (e.g. Wilmoth and Horiuchi 1999, Rossi et al. 2013). There is more spread for males than for females, especially since the mid-twentieth century. The HMD sample adds national populations over time, and, therefore, the increased spread over time for both sexes is partly an artifact of sample com-

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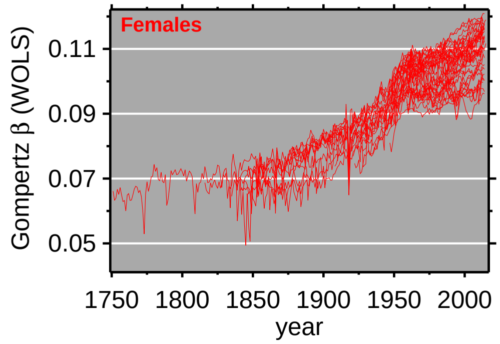

Figure 3: Evolution over time of Gompertzian slope parameter, $\hat{\beta}$, females. Shown are 41 countries, comprising 3,852 annual life tables.

position. Nonetheless, the comparison of spread between male and female still holds, because the HMD is balanced, sex-wise.

As is clear from the prediction equation, re-expressed as an equivalent product, $M_x = \exp(\alpha) \cdot \exp(\beta)^x$, mortality rates rise more steeply, the greater the value of β. Thus, a naïve interpretation of figures 2 (males) and 3 (females) might be that mortality rates above age 40 have been rising over time. Of course, it is well known that this is not the case; life expectancy at age 40 has been rising along with β, and a graph of e_{40} (not shown) would look much the same as figures 2 and 3. Indeed, the Pearson correlation between β and e_{40} in these data is 0.88 for males and 0.91 for females. This is not a paradox, but rather a simple case of baseline mortality (i.e., M_{40}) declining over the same period, such that it more than offsets increases in β. In other words, α sinks while β rises, such that e_{40} goes up all the while. Indeed, the Pearson correlation between empirical M_{40} and $\hat{\alpha}$ is 0.95 for males and 0.97 for females, even higher than that between $\hat{\beta}$ and e_{40}. This is not surprising; given the nature of the Gompertz model, α and M_{40} are related.

Figure 4 shows sex-specific scatterplots of the β vs. α estimates. As is evident from the figure, $\hat{\beta}$ and $\hat{\alpha}$ are very tightly (and negatively) related. The Pearson correlation is -0.983 for males, and -0.987 for females; as with all the correlations given, both are significant ($p < 0.00005$, but given the character of figure 4, the p-value is not what's important). As human mortality has changed over the last quarter millennium, the Gompertz β

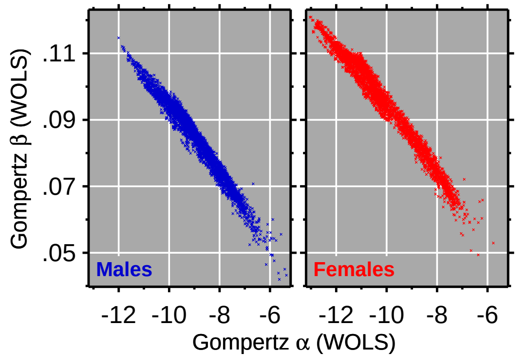

Figure 4: Scatterplot, $\hat{\beta}$ vs. $\hat{\alpha}$, 3,852 annual values per sex.

has increased, but the higher the β, the smaller the α. The net is more longevity—although changes in β may make it seem like old age mortality is more severe, it starts at a lower level. Changes in β alone are difficult to interpret.

The negative correlation in figure 4 is a classic finding, going back at least to Strehler and Mildvan (1960), who explain it as a prediction arising from a theory of aging in terms of another quantity, vitality, and an assumed Maxwell-Boltzmann distribution of energy expenditures. An alternate explanation for the negative correlation of α and β is in terms of compensation effects, in which M_x tends toward a common limit across populations as $x \to \omega$ (Gavrilov and Gavrilova, 1991, pp.148–156). Thus, lower starting points must be accompanied by higher slopes.

3.3 "Hand" and "thumb"

This analysis concerns age 40 and over. All the input data are from human populations in the age range, 40–100. However, the population at age 40 is not a *tabula rasa* so to speak, but is the result of the selection process of survival up to age 40 (Rohwer, 2016). This is different from analysis of life expectancy at birth, concerning mortality at all ages, in which the birth cohort is classically assumed to be a "clean slate". That is not fully correct, either, because there are substantial selection effects between conception and live birth (e.g. Catalano et al., 2015), and also social selection into pregnancy in different time periods (e.g. Currie and Schwandt, 2014).

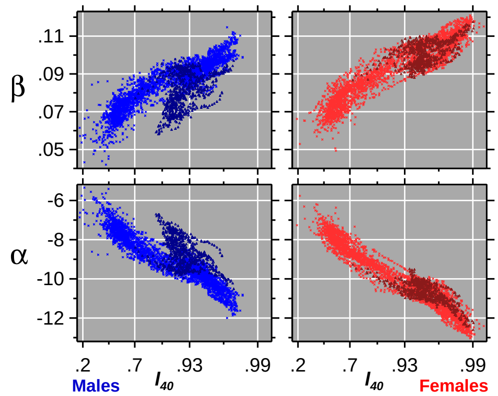

Figure 5: Scatterplots, $\hat{\beta}$ (top row) or $\hat{\alpha}$ (bottom row) vs. ℓ_{40}. 3,852 annual values per sex (males, left column; females, right column). The male graphs resemble a supinated/pronated hand, seen from the side (see main text). The "thumb" in the male graphs is caused by the formerly socialist economies of Europe, defined here as: BGR, BLR, CZE, DEUTE, EST, HUN, LTU, LVA, POL, RUS, SVK, SVN, UKR (see table 1 for abbreviations); these are plotted as triangles; all other countries as crosses. Horizontal axis $\log(-\log(\cdot))$ transformed.

Figure 5 shows sex-specific scatterplots of the α and β estimates vs. ℓ_{40}. The horizontal axis is transformed by $\log(-\log(\cdot))$, to linearize the relationship (Llewelyn, 1968). Given the tight and negative correlation between $\hat{\alpha}$ and $\hat{\beta}$ shown in figure 4, it is natural that the top row of figure 5 (β) is approximately the mirror image of the bottom row (α). When plotted against ℓ_{40}, a new feature emerges (for males but not for females), which is best described as a "thumb". This feature is composed of the formerly socialist economies (FSE) of Europe, listed in the figure 5 caption. These countries are plotted with a different symbol from the rest of the graph; the symbol difference may not be visible at this scale, but the thumb, as we call it, is extremely clear.

The thumb is a reflection of α values that are too high (or, equivalently, β values too low) for the corresponding ℓ_{40} value, relative to the main "hand" of the data. Recall that α is tightly related to M_{40} (in both theory and practice). Thus, these countries have mortality rates around age 40 which look high relative to integrated mortality prior to age 40 (i.e., relative to ℓ_{40}, which is $1 - {}_{40}q_0$, or one minus the life table probability of death from birth

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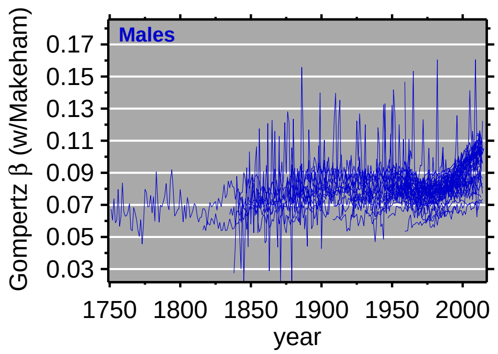

Figure 6: Evolution over time of Gompertzian slope parameter, $\hat{\beta}$, males, estimated using the Makeham model with Levenberg-Marquardt least squares. Shown are 41 countries, comprising 3,852 annual life tables.

to age 40). All else equal, the FSE countries perform poorly for mortality at ages $\geq$ 40—for males but not for females. Sex differences in mortality in the FSE have been observed before, with emphasis on Russia in particular (see, e.g., Shkolnikov et al. 1995, Gavrilova et al. 2000, Grigoriev et al. 2014, and Oksuzyan et al. 2014), but also in other countries (e.g., Jasilionis et al. 2011, Noymer and Van 2014). To the best of our knowledge, this has not been expressed before as a Gompertz parameter vs. ℓ_x relationship.

3.4 Makeham modification

An early modification of the Gompertz model is $M_x = \lambda + \alpha \exp(\beta x)$ (Makeham, 1860). Gompertz is special case of Makeham with $\lambda \equiv 0$. Estimating the Makheam parameter can be tricky (Feng et al., 2008); the log-transformed approaches that performed best, above, are not well suited to including λ as an additive offset in a one-pass approach. However, it is possible to estimate λ at the same time as α and β. At the suggestion of one of the anonymous referees, here we re-consider the graphical results when α and β have been co-estimated with λ. The point is that α and β change upon the inclusion of λ. The coefficients in this subsection were estimated using the L-M algorithm (Levenberg, 1944; Marquardt, 1963), with weights, using IDL 8.6.

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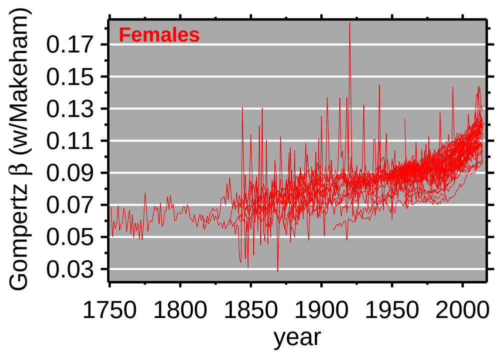

Figure 7: Evolution over time of Gompertzian slope parameter, $\hat{\beta}$, females. Shown are 41 countries, comprising 3,852 annual life tables.

Figure 6 shows the evolution over time of the β parameter when it is co-estimated with the Makeham offset, for males, and figure 6 is the same for females. These are analogous to figures 2 and 3, but when the estimation has been done with the Makeham term included. Omitting the Makeham term can potentially bias the estimates of β, since λ may decline over time, which may induce compensatory rises in β when the Makeham parameter is not part of the estimation (see Gavrilov and Gavrilova 1991, pp.143–6). Indeed, what figures 6 and 7 show is that, in the aggregate, the increase in β over time is less profound when it is estimated as part of a Makeham model.

Figure 8 shows the same relationship as figure 4, but estimated using the Levenberg-Marquardt (L-M) algorithm (but without λ). Note that the scale of the horizontal axis changes since $\exp(\alpha)$ from the WOLS approach is analogous to α from the unlogged approach. However, the qualitative agreement between figures 8 and 4 is evident, showing that this relationship is robust to using WOLS on the logged form, or using L-M nonlinear least squares on the unlogged form. Having shown that using L-M nonlinear least squares does not alter the basic relationship seen in figure 4, we can examine what happens to the same relationship when $\alpha, \beta, and\ \lambda$ are estimated simultaneously. This is shown in figure 9. The qualitative relationship between α and β is the same when these parameters are estimated in a Makeham versus a pure Gompertz model. However, the range of the parameters is greater in the Makeham model (i.e., comparing the vertical

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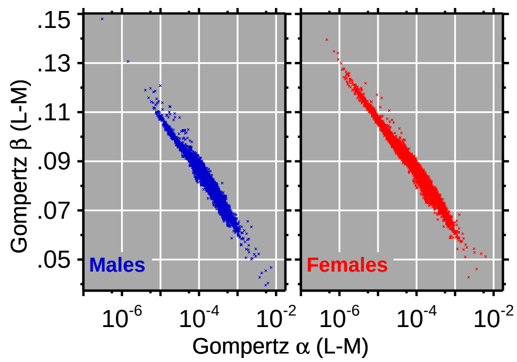

Figure 8: Scatterplot, $\hat{\beta}$ vs. $\hat{\alpha}$, 3,852 annual values per sex. Gompertz parameters estimated without the Makeham parameter, using the Levenberg-Marquardt algorithm.

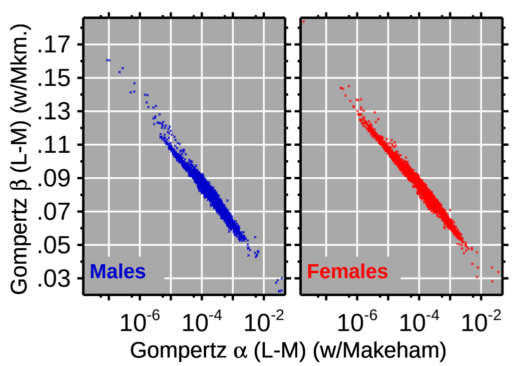

Figure 9: Scatterplot, $\hat{\beta}$ vs. $\hat{\alpha}$, 3,852 annual values per sex. Gompertz parameters estimated with the Makeham parameter, using the Levenberg-Marquardt algorithm.

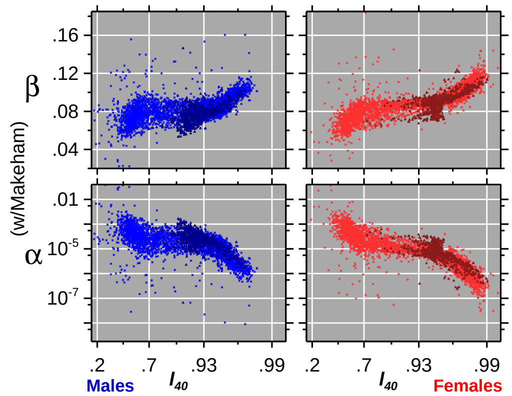

Figure 10: Replication of figure 5, but using Gompertz parameters in the presence of a Makeham offset (λ), estimated via the Levenberg-Marquardt algorithm.

axes).

Figure 10 is a replication of the "hand-thumb" graph, but, again, using Gompertz parameters that have been estimated in the presence of a Makeham offset, λ. Here, we see a qualitative distinction when compared to figure 5. Specifically, the thumb feature, although not invisible, is much less distinct when using the Gompertz coefficients estimated with the Makeham parameter. This is an important modification of the conclusion that the formerly socialist economies of Europe (as defined) have distinct male life tables compared to the rest of the HMD member countries. When the Makeham parameter is included, less so. Moreover, it should be noted that the value of the Gompertz coefficient versus $\log(-\log(\cdot))$ transformed ℓ_{40} graph as a tool of exploratory data analysis (in the sense of Tukey 1977) may depend on whether or not the model was estimated as Makeham or simple Gompertz.

4 Discussion

Much use of the Gompertz model in human demography concerns combining it with a frailty function to estimate the distribution of "robust" and "frail" individuals, net of observed mortality rates and assumptions about biological processes (among which, often that frailty is gamma-distributed).

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This literature is too large to survey here; for human mortality, the seminal article was Vaupel et al. (1979), and other examples include Manton and Stallard (1984), Vaupel and Yashin (1985), Manton et al. (1986), Weiss (1990), Yashin and Iachine (1997), Yashin et al. (2002), and Wienke et al. (2003). The literature is likewise vast for other species, e.g., Rose (1991), Pletcher et al. (2000), and Carey (2003). Other approaches to the problem are exemplified by Gavrilov and Gavrilova (2001).

The frailty approach offers another explanation for the negative correlation seen in figure 4. As mortality falls over time, it tends to do so across the life course (i.e., comparing one life table to another, not from the point of view of an individual, who, after childhood, will experience rising mortality with age). Lower-mortality populations have greater ℓ_{40}, so have been subject to less negative selection up to age 40, in the frail-robust framework. Although α is lower in such populations, they are primed to have a higher β because, having a greater proportion frail at age 40, there is more scope for mortality selection above age 40. This phenomenon is sometimes called cohort inversion (Hobcraft et al., 1982).

This is especially relevant to figure 5, where we see the predicted relationship: positive correlation between ℓ_{40} and β. However, the presence of the previously-described thumb shows that the slope of this scatterpot is not something that is hard-wired into human populations, but is socially malleable. The constituent countries of the thumb form a coherent group, the formerly socialist economies of Europe. Moreover, the thumb is exclusively a male phenomenon. This points to environmental (including social) causes, and, indeed, male behavior in these countries, especially as regards alcohol, is a well-understood problem (Bhattacharya et al. 2013, Zaridze et al. 2014). The HMD is not a representative sample of world populations, with Africa not represented at all, and Asia and Latin America underrepresented. It is therefore an open question whether additional thumbs exist in human populations.

There are many extensions of the Gompertz model. The Heligman-Pollard (1980) model (cf. also Thatcher 1990), and the Kannisto, or logistic, hazard model (Thatcher et al., 1998) are two of the best known. These models assume that $\frac{d^2}{dx^2}M_x < 0$ when $x \gtrsim 90$, but have a continuous and always-positive first derivative, so that the increase in the risk of death slows down at oldest-old ages (Horiuchi and Wilmoth, 1998). The life table aging rate (see Horiuchi and Wilmoth 1997) is also an important approach to this phenomenon. Gage (1989) reviews models which join the Gompertz relationship to functions for morality earlier in life.

Despite these elaborations and extensions, the "plain" Gompertz model remains a workhorse of practical demography. Applications include closing out life tables—viz., imputing m_x (or M_x) at ages for which the input data are noisy (Coale and Demeny 1983, see also Coale and Guo 1989). It bears repetition that the empirical pattern of human mortality is Gompertzian from around age 40 to around age 95. The frailty approach is an attempt to understand the distribution of unobserved heterogeneity (e.g., Zajacova et al., 2009), net of observed M_x. Deviations from Gompertz hazards among centenarians, which start somewhere in 90s of age (or even later, Gavrilov and Gavrilova 2011), are a fascinating aspect of of human mortality (Kannisto, 1988), but even today apply only to a small fraction of the population. Gompertzian mortality is an important relationship in formal demography, and identifying best-practices for its estimation is clearly a desideratum for human population studies (and, perhaps, for other species as well, e.g., Gavrilova and Gavrilov 2015).

The main goal of this paper was to determine the best Gompertz fitting procedure among a portfolio of practical options. The answer is to use weighted least squares regression of logged death rates (WOLS), using the number of deaths for each observation as the weights. However, it must be noted that WOLS is not always the best model; by the e_{40} criterion, Poisson regression performs better in 996 cases for males and 1,027 cases for females, out of 3,852 populations per sex. Another possibility would be to run both WOLS and Poisson regression, then to select the Gompertz coefficients of the better-fitting model. This is not what we did to produce the graphs, since there is something to be said for consistency. However, the M_x prediction equations for WOLS and Poisson are the same, so choosing the best-fitting coefficients by a compound estimating technique can be justified. However, this approach would be considerably more laborious, since it would require calculating e_{40} for two candidate models and the raw data, prior to choosing the final model. In cases where death numbers are unavailable (for instance, with a published table of death rates, only), we recommend estimating Gompertz parameters by regression of logged death rates on age.

The novel contribution of this paper is that we ran a "tournament" of five possible ways of estimating empirical Gompertz, using all available data from the Human Mortality Database. Weighted ordinary least squares regression is the best-practices approach to Gompertz parameter estimation in cases where there is no *a priori* reason to do it another way. Our presentation of the tight and negative correlation of α and β (figure 4) is not a novel finding. Visualizing Gompertz parameter trends in the former socialist economies of Europe as a thumb (figure 5) is new as far as we are aware. It is consistent with existing subject matter knowledge in European mortality studies. The $\log(-\log(\ell_x))$ transformation (also suggested by Thatcher 1990, p.142), as in figure 5, probably deserves more use in population studies.

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